What is torsion?

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Let me begin by saying that torsion has not yet been measured, so perhaps it is not surprising there is more than one theory of torsion. I will describe torsion of the string theory kind. This field may be described, like electromagnetism (E&M), with the use of vector calculus. In the following description I will stress the similarities, and differences, between torsion and electro-magnetism (and so I recommend you read this after you had the junior level course in electromagnetism) Also, I will begin with basic notions of vectors and scalars as is used in E&M course. Later I will show how this can be formulated in what we call covariant form, where we use four-vectors. Just about everything I will describe here has been published, and will cite the literature as I go. Also I will provide motive and rational for the following later, but let's not dawdle any longer.

mathematical formulation

First off, the field is described by two objects, a vector and a scalar, which are denoted by $\mathbf{b}$ and ϕ. I will also mention that the formulation you are about see has been published.[1] As in E&M, the field is described in terms of a potential. Unlike E&M there are two vector potentials ($\boldsymbol{A}$) and ($\boldsymbol{a}$). From these the field is defined as

$$\phi = \boldsymbol{\nabla} \cdot \boldsymbol{a}, \tag{1}$$

$$\boldsymbol{b} = \boldsymbol{\nabla} \times \boldsymbol{A} - \dot{\boldsymbol{a}} \tag{2}$$

where the overdot means partial differentiation with respect to ct, which is simply $1/c$ times the partial derivative with respect to time. One should note: Take the divergence of (2) and the partial time derivative of (1) with respect to time. Using the fact the divergence of the curl is zero we have,

$$\boldsymbol{\nabla} \cdot \boldsymbol{b} - \frac{1}{c}\frac{\partial \phi}{\partial t} = 0, \tag{3}$$

which is an identity.

Like E&M, torsion is gauge invariant. For any scalar λ and any vector $\boldsymbol{V}$, if you let

$$\begin{aligned} \boldsymbol{a} &\rightarrow \boldsymbol{a} + \boldsymbol{\nabla} \times \boldsymbol{V} \\ \boldsymbol{A} &\rightarrow \boldsymbol{A} + \boldsymbol{\nabla}\lambda + \dot{\boldsymbol{V}}, \end{aligned} \tag{4}$$

then equations (1) and (2) do not change at all. I will note that gauge invariance is like exercise, the older you get the more important it becomes.

In E&M there are four components to the potential (three from the vector potential and one from the scalar) which correspond to four components of the source (three for the vector current density and one for the charge density). The same relation holds with torsion, and since there are six components to the potential (two vectors) there are six components to the source. These are represented by two vectors $\boldsymbol{N}$ and $\boldsymbol{I}$. With these the field equations are

$$\boldsymbol{\nabla}\phi = -\dot{\boldsymbol{b}} - \boldsymbol{I} \tag{5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{b} = \boldsymbol{N}. \tag{6}$$

These represent torsion just as Maxwell equations represent electromagnetism.

Let us take stock. Torsion is described by the vector $\boldsymbol{b}$ and the scalar ϕ. These quantites may be derived from a potential, and are coupled to the source as shown above. The results are collected and compared to E&M in Figs. I and II.

physical meaning

So far all we have is a bunch of mathematical equations. Suppose I ask you, "Now that you understand the previous section, how would you go about measuring torsion?" That's a hard question, especially since I have not given a physical interpretation to the source vectors $\boldsymbol{N}$

	Torsion	E&M
Potential	$\boldsymbol{a}$, $\boldsymbol{A}$	ϕ, $\boldsymbol{A}$
Field from potential	$\boldsymbol{b} = \boldsymbol{\nabla} \times \boldsymbol{A} - \dot{\boldsymbol{a}}$ $\phi = \boldsymbol{\nabla} \cdot \boldsymbol{a}$	$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \dot{\boldsymbol{A}}$ $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$
Gauge Invariance	$\boldsymbol{a} \to \boldsymbol{a} + \boldsymbol{\nabla} \times \boldsymbol{V}$ $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\lambda + \dot{\boldsymbol{V}}$	$\lambda \to \lambda - \dot{\lambda}$ $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\lambda$

FIG. 1: Comparison of fields from a potential and gauge invariance

	Torsion	Electromagnetism
Field Equations	$\boldsymbol{\nabla}\phi = -\dot{\boldsymbol{b}} - \boldsymbol{I}$ $\boldsymbol{\nabla} \times \boldsymbol{b} = \boldsymbol{N}$	$\boldsymbol{\nabla} \times \boldsymbol{H} = 4\pi \boldsymbol{J} + \dot{\boldsymbol{D}}$ $\boldsymbol{\nabla} \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0$ $\boldsymbol{\nabla} \cdot \boldsymbol{D} = 4\pi\rho$ $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$
Potential	$\boldsymbol{A} = q \int \frac{\boldsymbol{\sigma}}{\lvert \boldsymbol{r}-\boldsymbol{r}' \rvert} dV'$	$\boldsymbol{A} = \frac{1}{c} \int \frac{\boldsymbol{J}}{\lvert \boldsymbol{r}-\boldsymbol{r}' \rvert} dV'$

FIG. 2: The field equations and potential in terms of the source for electromagnetism and torsion. σ is related to the intrinsic spin and will be given in detail in the more advanced sections.

and $\boldsymbol{I}$.

As is mentioned before, my job here is to make you understand torsion, the motivations and beauty will all be demonstrated later. The reason I am presenting the material in this fashion because most other descriptions do it in the opposite fashion, and many readers get hopelessly lost long before the equations of torsion are studied.

So, I will continue in this fashion and simply tell you the source vectors are related to spin (I will prove it later). In the rest frame of the particle (producing torsion), $\boldsymbol{I}$ is zero. In the frame in which the particle is moving, $\boldsymbol{I}$ enters with a factor of v/c where v is the velocity, so let's ignore it for now. That leaves torsion created by the source N, which is essentially the intrinsic spin.

So once again let us take stock. We have the intrinsic spin described by the source vector $\boldsymbol{N}$. This creates a field according to the equations given above. This is like saying we have charge described by $\boldsymbol{j}$ and ρ and the field is given by Maxwell's equations.

Now you know what torsion is, a field produced by intrinsic spin and described by the mathematical equations given above. But let's get a little more concrete and look at a specific solution. We consider we are in the rest frame of a particle with intrinsic spin $\boldsymbol{S}$.

Personal anecdote

Years ago, just after I formulated this theory of torsion in curved space, I set about trying to find out the physical significance. (Prior to that I investigated using torsion as an electromagnetic potential to geometrize electromagnetism, but I abandoned that.) Others were claiming torsion is related to spin, and there were many other interpretations. I was happy with my mathematical formulation of torsion, but was careful not to impose my beliefs or hopes on what torsion is. Let the equations tell you what they represent.

And so, I searched for a spherically symmetric solution. (For example Schwarschild found his eponymous solution to Einstein's field equations and it gave us great insight into gravity, about black holes, and so on.) I worked on this for months, but every time I thought I was getting close to a solution, something popped up that disallowed the solution. Then it finally hit me, what if spherical solutions do not exist? It took me two days to show static spherically solutions did not exist,[2] but then I got worried.

I said to myself, spherically symmetric solutions exist for gravity, spherically symmetric solutions exist for E&M, why not for torsion? Can a field exist that does have spherically symmetric solutions? Maybe my formulation is wrong after all.

That is when I realized that torsion might result from spin after all. Unlike mass and charge, intrinsic spin is a vector. It therefore establishes a direction in space. Therefore spherically symmetric solutions cannot exist, and I was happy once again.

back to business

Suppose we consider a multipole expansion of the source, just as we do in E&M.

$$A_n = \frac{1}{c|\boldsymbol{x}|} \int J_n dV' + \frac{|\boldsymbol{x}|}{c|\boldsymbol{x}|^3} \cdot \int J_n \boldsymbol{x}' dV' + \ldots \quad (7)$$

where $\boldsymbol{x}$ goes from the origin to the point at which the field is evaluated and $\boldsymbol{x}'$ goes from the origin to a point in the source.The first term is symmetric and so must be thrown out, leaving

$$\boldsymbol{A} = \frac{\boldsymbol{\mu} \times \boldsymbol{x}}{x^3} \quad (8)$$

where $x = |\boldsymbol{x}|$. Then, from (2), we have

$$\boldsymbol{b} = \kappa \frac{6G}{c^3} \boldsymbol{S} r^3 (2\cos\theta \hat{\boldsymbol{r}} + \sin\theta \hat{\boldsymbol{\theta}}) \tag{9}$$

where κ is the dimensioless coupling constant (to be measured) and where, as stated above $\boldsymbol{S}$ is the intrinsic spin. One should note that, since we are solving a first order differential equation there is one unknown constant. The factors appearing in (9) may be all combined into a single constant, but it is written this way so that we have a non-dimensional coupling constant.

The interaction

As I mentioned earlier, this entire theory was formulated in curved space as a generalized theory of gravitation. One good thing about general relativity (gravitation) is that you do not have to postulate the equation of motion, it follows from the field equation. In contrast, in Newtonian gravity you must postulate the field equation, and then in addition you must postulate the equation of motion (F=ma). Why did I call it a good thing? Because it leads to testable consequences. If you could separately postulate and equation of motion, then you cannot be sure your theory is right. In fact, you can't be sure of anything.

And so, I must borrow the result I found from general relativity, to be described in the more advanced sections. The result is that the potential energy U of a spin $\boldsymbol{S}$ in a torsion field $\boldsymbol{b}$ is (**wait** try and guess the answer)

$$U = \frac{c}{2} \kappa \boldsymbol{b} \cdot \boldsymbol{S}. \tag{10}$$

Now you can answer my question, how do you measure torsion, by looking for the interation described by (10).

This is section is now ending. I hope you understand more about torsion than you did before you read this. The next section is written more at the graduate student level, and I hope you will continue reading (if not now then later).

The reason there is only one citation that shows the vector formulation is because there is only one reference. However, if you would like to see an introduction to torsion as it appears as part of general relativity that is not written to the experts, see [3].

Although the references below are all by me, there are many people who study torsion. A long list is given in bibliography of a technical review I wrote.[4]

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[1] R. T. Hammond, "Spin, the classical theory," J. Mod. Phys., 1 **3** (2012).
[2] R. T. Hammond, "Non-existence of static, spherically symmetric solutions with torsion potential," Class. Quantum Grav., 8, L175 (1991).
[3] R. T. Hammond, "New Fields in General Relativity," Cont. Phys. **36**, 103 (1995).
[4] R. T. Hammond, "Torsion Gravity," Reports on Progress in Physics, **65**, 599 (2002).